EXERCISE 1(a) Factorize the joint density:

$$f(x_1,\dots,x_n) = \theta^n \cdot e^{-\theta \sum\limits_{i=1}^{n} x_i} = g(\sum_{i=1}^{n} x_i, \theta) \cdot h(x_1,\dots,x_n)$$

with $g(x,\theta) = \theta^n \cdot e^{-\theta \cdot x}$ and $h(x_1,\dots,x_n) = 1$.
It follows from the factorization theorem that $\sum\limits_{i=1}^{n} x_i$ is a sufficient statistic.

1(b) Method of Moments (MM) estimator:

$$\bar{X} = E[X] = \frac{1}{\theta} \quad \Leftrightarrow \quad \hat{\theta}_{MM} = \frac{1}{\bar{x}} = \frac{n}{\sum\limits_{i=1}^{n} X_i}$$

and Maximum Likelihood (ML) estimator:

$$\begin{aligned} l_X(\theta) = \log\left(\theta^n \cdot e^{-\theta \sum\limits_{i=1}^{n} X_i}\right) &= n\log(\theta) - \theta \sum_{i=1}^{n} X_i \\ \frac{d}{d\theta}\, l_X(\theta) &= \frac{n}{\theta} - \sum_{i=1}^{n} X_i \\ \frac{d^2}{d\theta^2}\, l_X(\theta) &= -\frac{n}{\theta^2} \end{aligned}$$

Set the 1st derivative equal to zero: $\frac{n}{\theta} - \sum\limits_{i=1}^{n} X_i = 0 \quad \Leftrightarrow \quad \hat{\theta}_{ML} = \frac{n}{\sum\limits_{i=1}^{n} X_i}$.
And this is a maximum point, since $\frac{d^2}{d\theta^2}\, l_X(\theta) = -\frac{n}{\theta^2} < 0$ for all $\theta > 0$.

1(c) We have:

$$I(\theta) = -E\left[\frac{d^2}{d\theta^2} l_{X_1}(\theta)\right] = -E\left[-\frac{1}{\theta^2}\right] = \frac{1}{\theta^2}$$

1(d) We have:

$$\sqrt{n} \cdot (\hat{\theta}_{ML} - \theta) \xrightarrow{D} \mathcal{N}\left(0, \frac{1}{I(\theta)}\right) = \mathcal{N}\left(0, \theta^2\right) \quad \Leftrightarrow \quad \text{for large n: } \hat{\theta}_{ML} \sim \mathcal{N}\left(\theta, \frac{\theta^2}{n}\right)$$

1(e) For large n we have: $\sqrt{n} \cdot \sqrt{I(\theta)} \cdot (\hat{\theta}_{ML} - \theta) \sim \mathcal{N}\left(0, 1\right)$. Therefore:

$$\begin{aligned} P(-1.64 \le \sqrt{n} \cdot \sqrt{I(\theta)} \cdot (\hat{\theta}_{ML} - \theta) \le 1.64) &= 0.9 \\ \Leftrightarrow P\left(\hat{\theta}_{ML} - \frac{1.64}{\sqrt{n} \cdot \sqrt{I(\theta)}} \le \theta \le \hat{\theta}_{ML} + \frac{1.64}{\sqrt{n} \cdot \sqrt{I(\theta)}}\right) &= 0.9 \end{aligned}$$

Using the observed Fisher information, the 90% CI is given by

$$\hat{\theta}_{ML} \pm \frac{1.64}{\sqrt{n} \cdot \sqrt{I(\hat{\theta}_{ML})}}$$

We have $\bar{X}_{25} = 4 \Leftrightarrow \hat{\theta}_{ML} = 1/4 = 0.25$, so that we get:

$$0.25 \pm \frac{1.64}{\sqrt{25} \cdot \sqrt{\frac{1}{0.25^2}}} \quad \Rightarrow \quad 0.25 \pm \frac{1.64}{20} \quad \Rightarrow \quad [0.168; 0.332]$$

1(f) Under H_0 we have for large n: $\sqrt{25} \cdot \sqrt{I(0.5)} \cdot (\hat{\theta}_{ML} - 0.5) \sim \mathcal{N}(0, 1)$. Therefore:

$$\begin{aligned}
P(-1.96 \leq 10 \cdot (\hat{\theta}_{ML} - 0.5) \leq 1.96) &= 0.9 \\
\Leftrightarrow P\left(0.5 - \frac{1.96}{10} \leq \hat{\theta}_{ML} \leq 0.5 + \frac{1.96}{10}\right) &= 0.9 \\
\Leftrightarrow P\left(0.304 \leq \frac{1}{\bar{X}_{25}} \leq 0.696\right) &= 0.9 \\
\Leftrightarrow P\left(\frac{1}{0.304} \geq \bar{X}_{25} \geq \frac{1}{0.696}\right) &= 0.9
\end{aligned}$$

And so H_0 can be rejected if $\bar{X}_{25} \geq 3.29$ or $\bar{X}_{25} \leq 1.44$, i.e. if $\bar{X}_{25} \notin [1.44; 3.29]$.

1(g) For the α quantile q_α we have: $F(q_\alpha) = \alpha$. Solve the equation for q_α:

$$F(q_\alpha) = 1 - e^{-\theta q_\alpha} = \alpha \quad \Leftrightarrow \quad q_\alpha = -\frac{\log(1-\alpha)}{\theta}$$

1(h) It follows from (g):

$$\begin{aligned}
P\left(-\frac{\log(1-0.05)}{\theta} \leq X_1 \leq -\frac{\log(1-0.95)}{\theta}\right) &= 0.9 \\
\Leftrightarrow P\left(-\frac{\log(0.95)}{X_1} \leq \theta \leq -\frac{\log(0.05)}{X_1}\right) &= 0.9
\end{aligned}$$

For $X_1 = 4$ we get: $P(0.013 \leq \theta \leq 0.749) = 0.9$, so that the exact 90% CI is $[0.013, 0.749]$.

EXERCISE 2(a) For $\mu = 0$ the joint density is:

$$\begin{aligned} f(x_1,\dots,x_n) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \cdot \exp\left\{-\frac{1}{2} \cdot \frac{x_i^2}{\sigma^2}\right\} \\ &= (2\pi)^{-n/2} \cdot \left(\sigma^2\right)^{-n/2} \cdot \exp\left\{-\frac{1}{2} \cdot \frac{\sum_{i=1}^{n} x_i^2}{\sigma^2}\right\} \end{aligned}$$

This can be factorized into:

$$f(x_1,\dots,x_n) = g(\sum_{i=1}^{n} x_i^2, \sigma^2) \cdot h(x_1,\dots,x_n)$$

with $g(x,\sigma^2) = (2\pi)^{-n/2} \cdot (\sigma^2)^{-n/2} \cdot \exp\left\{-\frac{1}{2} \cdot \frac{x}{\sigma^2}\right\}$ and $h(x_1,\dots,x_n) = 1$.
It follows from the factorization theorem that $\sum_{i=1}^{n} x_i^2$ is a sufficient statistic.

2(b) The log likelihood is:

$$l_X(\sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\left(\sigma^2\right) - \frac{1}{2} \cdot \frac{\sum_{i=1}^{n} x_i^2}{\sigma^2}$$

The 1st derivative is:

$$\frac{d}{d\sigma^2}\, l_X(\sigma^2) = -\frac{n}{2}\left(\frac{1}{\sigma^2}\right) + \frac{1}{2} \cdot \frac{\sum_{i=1}^{n} x_i^2}{\sigma^4}$$

Setting the 1st derivative to zero, yields:

$$-\frac{n}{2}\left(\frac{1}{\sigma^2}\right) + \frac{1}{2} \cdot \frac{\sum_{i=1}^{n} x_i^2}{\sigma^4} = 0 \;\Leftrightarrow\; \frac{1}{2} \cdot \frac{\sum_{i=1}^{n} x_i^2}{\sigma^2} = \frac{n}{2} \;\Leftrightarrow\; \sigma^2_{ML} = \frac{\sum_{i=1}^{n} x_i^2}{n}$$

2nd derivative check: Compute the 2nd derivative w.r.t. σ^2:

$$\frac{d^2}{d(\sigma^2)^2}\, l_X(\sigma^2) = -\frac{n}{2}\left(\frac{1}{(\sigma^2)^2}\right) \cdot (-1) + \frac{1}{2} \cdot \frac{\sum_{i=1}^{n} x_i^2}{(\sigma^2)^3} \cdot (-2) \;=\; \frac{n}{2}\left(\frac{1}{(\sigma^2)^2}\right) - \frac{\sum_{i=1}^{n} x_i^2}{(\sigma^2)^3}$$

Plugging-in $\sigma^2_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i^2$ yields:

$$\frac{n}{2}\left(\frac{n^2}{\left(\sum_{i=1}^{n} x_i^2\right)^2}\right) - \frac{n^3 \cdot \sum_{i=1}^{n} x_i^2}{\left(\sum_{i=1}^{n} x_i^2\right)^3} = \left(\frac{\frac{1}{2}n^3 - n^3}{\left(\sum_{i=1}^{n} x_i^2\right)^2}\right) = -\left(\frac{\frac{1}{2}}{\left(\sum_{i=1}^{n} x_i^2\right)^2}\right) < 0$$

so that σ^2_{ML} is indeed a maximum point.

2(c) Compute the LR:

$$W(X_1,\dots,X_n) \;=\; \frac{(2\pi)^{-n/2}\cdot 1^{-n/2}\cdot \exp\left\{-\frac{1}{2}\cdot\frac{\sum_{i=1}^{n} X_i^2}{1}\right\}}{(2\pi)^{-n/2}\cdot 4^{-n/2}\cdot \exp\left\{-\frac{1}{2}\cdot\frac{\sum_{i=1}^{n} X_i^2}{4}\right\}} = 4^{n/2}\cdot\exp\left\{-\frac{3}{8}\sum_{i=1}^{n} X_i^2\right\}$$

and in outlook to part (d) we note that this is a monotonically decreasing function in $\sum_{i=1}^{n} X_i^2$.

2(d) We reject H_0 if $W(X_1,\dots,X_5) < c$, and we have:

$$W(X_1,\dots,X_5) < c \;\Leftrightarrow \sum_{i=1}^{5} X_i^2 > k$$

Under H_0 we have $\sum_{i=1}^{5} X_i^2 \sim \chi_5^2$, so that k must be the 0.95 quantile of the χ_5^2 distribution.

Hence, the UMP rejects H_0 if $\sum_{i=1}^{5} X_i^2 > 11.07$.

The realization $\sum_{i=1}^{5} X_i^2 = 10 < 11.07$ would thus not lead to a rejection of H_0.

2(e) Under $H_1: \sigma^2 = 4$ we have: $\sum_{i=1}^{5}\left(\frac{X_i}{2}\right)^2 = \sum_{i=1}^{5}\frac{X_i^2}{4} \sim \chi_5^2$, and moreover $P\left(\sum_{i=1}^{5} X_i^2 > 11.07\right) \Leftrightarrow P\left(\sum_{i=1}^{5}\frac{X_i^2}{4} > 2.77\right)$.

So the power at $\sigma^2 = 4$ is the probability that a χ_5^2 distribution takes a value larger than 2.77. And from the quantile table it can be seen:

- The probability that a χ_5^2 distribution takes a value larger than $q_{0.1} = 1.61$ is 0.9.
- The probability that a χ_5^2 distribution takes a value larger than $q_{0.5} = 4.35$ is 0.5.

Hence, the probability that a χ_5^2 distribution takes a value larger than 2.77 (=the power) must be in between 0.5 and 0.9.

2(f) We have:

$$W(X_1,\dots,X_5) = 0.3 \;\Leftrightarrow\; 4^{5/2}\cdot\exp\left\{-\frac{3}{8}\sum_{i=1}^{5} X_i^2\right\} = 0.3 \;\Leftrightarrow\; \sum_{i=1}^{5} X_i^2 = -\frac{8}{3}\log\left(\frac{0.3}{2^5}\right) = 12.45$$

As $12.45 > 11.07$ (cf. part (d)), H_0 could be rejected to the level $\alpha = 0.05$.

2(g) We have

$$\lambda(X_1,\ldots,X_n) = \frac{(2\pi)^{-n/2}\cdot 1^{-n/2}\cdot \exp\left\{-\frac{1}{2}\cdot\frac{\sum_{i=1}^n X_i^2}{1}\right\}}{(2\pi)^{-n/2}\cdot(\hat{\sigma}^2_{ML})^{-n/2}\cdot\exp\left\{-\frac{1}{2}\cdot\frac{\sum_{i=1}^n X_i^2}{\hat{\sigma}^2_{ML}}\right\}}$$

$$= \{\hat{\sigma}^2_{ML}\}^{n/2}\cdot\exp\left\{-\frac{1}{2}\sum_{i=1}^n X_i^2+\frac{1}{2}n\right\}$$

$$= \{\hat{\sigma}^2_{ML}\}^{\frac{n}{2}}\cdot\exp\left\{-\frac{n}{2}\cdot\hat{\sigma}^2_{ML}+\frac{n}{2}\right\}$$

(Though not asked for, we note: $a := n/2$.)

2(h) As $e^{n/2}$ is only a positive factor, we have the relationship:

$$\{\hat{\sigma}^2_{ML}\}^{\frac{n}{2}}\cdot\exp\left\{-\frac{n}{2}\cdot\hat{\sigma}^2_{ML}+\frac{n}{2}\right\}<c \quad\Leftrightarrow\quad \{\hat{\sigma}^2_{ML}\}^{\frac{n}{2}}\cdot\exp\left\{-\frac{n}{2}\cdot\hat{\sigma}^2_{ML}\right\}<c_0$$

We would like to reject $H_0:\sigma^2=1$ if $\hat{\sigma}^2_{ML}$ takes either a very low or a very high realization.

According to hint (3), we have:

- For $\hat{\sigma}^2_{ML}<1$:

$$\{\hat{\sigma}^2_{ML}\}^{\frac{n}{2}}\cdot\exp\left\{-\frac{n}{2}\cdot\hat{\sigma}^2_{ML}\right\}<c_0 \quad\Leftrightarrow\quad \hat{\sigma}^2_{ML}<k \quad\Leftrightarrow\quad \sum_{i=1}^n X_i^2<k_0$$

- For $\hat{\sigma}^2_{ML}>1$:

$$\{\hat{\sigma}^2_{ML}\}^{\frac{n}{2}}\cdot\exp\left\{-\frac{n}{2}\cdot\hat{\sigma}^2_{ML}\right\}<c_0 \quad\Leftrightarrow\quad \hat{\sigma}^2_{ML}>k^\star \quad\Leftrightarrow\quad \sum_{i=1}^n X_i^2>k_0^\star$$

Under H_0 we have: $\sum_{i=1}^n X_i^2\sim\chi^2_n$, so we select

- k_0 is the 0.05 quantile of the χ^2_n distribution.
- $k_0^\star$ is the 0.95 quantile of the χ^2_n distribution.

For $n=5$ that means we reject H_0 if

- either $\sum_{i=1}^5 X_i^2<q_{0.05}=1.15$
- or $\sum_{i=1}^5 X_i^2>q_{0.95}=11.07$

In other words:
We reject H_0 if the realization of $\sum_{i=1}^5 X_i^2$ is outside the interval $[1.15;11.07]$.